Random walk on a linear chain with a quenched distribution of jump lengths

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We study the random walk of a particle on a linear chain, where a jump length 1 or 2 is assigned randomly to each lattice site with probability p_1 and $p_2=1-p_1$, respectively. We find that the probability $p_1^{\rm eff}$ for the particle to be at a site with jump length 1 is different from p_1, which causes the diffusion coefficient D to differ from the mean-field result. A theory is developed that allows us to calculate $p_1^{\rm eff}$ and D for all values of p_1. In the limit $p_1\to 0$, the theory yields a nonanalytic dependence of $p_1^{\rm eff}$ on p_1, $p_1^{\rm eff}\sim -p_1^2\ln p_1$.
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I. INTRODUCTION

Random walks in disordered systems have attracted much attention in the past due to their importance in many physical, chemical, and biological processes (for reviews see [1–3]). In lattice models one usually considers one particle that can hop among nearest-neighbor sites with random transition rates. These transition rates may fluctuate in time independent of the position of the walker (''annealed disorder'') or they may be assigned randomly to the sites or bonds of the lattice independent of time (''quenched disorder''). The latter situation is more natural for most applications; e.g., the quenched transition rates may arise from a random energy landscape or some topological disorder present in the material. More generally, one could consider the random walk in a dynamic random structure, whose properties change in a characteristic time τ_s. Then a crossover in the diffusion properties occurs from the behavior in the quenched case for times $t\ll\tau_s$ to the behavior in the annealed case for times $t\gg\tau_s$. Both the annealed and quenched cases have been investigated in detail and most of the important results can be found in [1–3].

When the random walker is allowed to jump farther than to nearest-neighbor sites according to some distribution p_l of jump lengths l, one may again distinguish between the annealed and the quenched situation: In the first case, the jump lengths vary in time independently of the position of the walker, while in the latter they are fixed to the lattice sites. The annealed case usually presents no additional difficulty and can be studied within a standard mean-field treatment [4]. The quenched problem, however, is much more subtle and so far has not been studied. This is certainly due to the fact that a random walk with a quenched jump lengths distribution is not a very realistic model to describe particle diffusion in a random potential. However, it may be important in other situations such as, e.g., to describe tracer diffusion in turbulent fluids by using the concept of Lévy walks [5] or in recent experiments where chaotic transport is observed in two-dimensional flow in a rapidly rotating annular tank [6].

We will show in this work that already the simplest case, i.e., a random walk on a linear chain, where the walker can jump either to nearest- (jump length 1) or to next-nearest-neighbor sites (jump length 2), exhibits strong deviations from a simple mean-field approach. These deviations are caused by two effects: (i) The random walker stays longer in clusters consisting of lattice sites with jump length 1 than in clusters of the same size consisting of lattice sites with jump length 2 and (ii) isolated lattice sites with jump length 1, whose nearest neighbors have jump length 2, can be overjumped.

II. A SIMPLE RANDOM-WALK MODEL

Consider the random walk of one particle on a linear chain as shown in Fig. 1. To each lattice site i a jump length l_i is assigned, which can assume two possible values, $l_i=1$ or $l_i=2$ with probabilities p_1 and $p_2=1-p_1$, respectively. We will call a site i a 1 site if $l_i=1$ and a 2 site otherwise. If the walker stays at a 1 site, it can jump to one of the two nearest-neighbor sites, while if it stays at a 2 site, it can jump to one of the two next-nearest-neighbor sites. The time between two jumps is distributed according to a Poisson distribution with mean waiting time τ. We will work with discrete time steps, choosing τ as time unit. Also, we will restrict our treatment to the unbiased case, i.e., in each time step $\tau=1$ the walker jumps with equal probability 1/2 to the left or to the right. A possible path for the walker is shown in Fig. 1. Note that only the jump lengths are quenched but not the jump directions (in this case the motion would be deterministic and the particle gets trapped).

The probability $P_i(t)$ that the walker is at site i at time t obeys the recursion relation

$$P_i(t+1)=\frac{1}{2}[(l_{i-2}-1)P_{i-2}(t)+(2-l_{i-1})P_{i-1}(t)+(2-l_{i+1})P_{i+1}(t)+(l_{i+2}-1)P_{i+2}(t)], \tag{1}$$

which has to be solved subject to some given initial condition. We are interested in the diffusion properties in the stationary state. In a mean-field approach, one would substitute the individual l_i in Eq. (1) by their average value $\langle l_i\rangle=p_1+2p_2=2-p_1$ and the mean-square displacement of

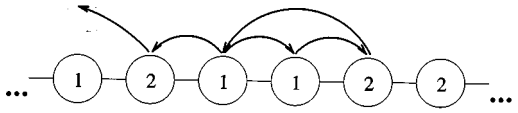

FIG. 1. Linear chain, where jump lengths 1 and 2 are assigned randomly to the lattice sites. The arrows indicate a possible diffusion path for the random walker.

the walker would be given by $\langle x^2(t)\rangle=\langle l_i^2\rangle t=(p_1+4p_2)t=(4-3p_1)t$. Accordingly, we obtain for the mean-field diffusion coefficient

$$D_{\mathrm{MF}}=\frac{4-3p_1}{2}. \tag{2}$$

III. DEVIATIONS FROM THE MEAN-FIELD RESULT

In order to test up to what extent Eq. (2) gives a good approximation we have determined the mean-square displacement $\langle x^2(t)\rangle$ by Monte Carlo simulations [7]. When the walker is allowed to make a few hundred jumps before its initial position its stored (i.e., after ''equilibration''), we find that $\langle x^2(t)\rangle=2Dt$ for all times t. The diffusion coefficient $D(p_1)$ as a function of p_1 is shown in Fig. 2 (open circles) together with the mean-field result $D_{\mathrm{MF}}(p_1)$ (dashed line). As can be seen from the figure, D is larger than D_{MF} for small values $p_1\lesssim 0.22$, while for $p_1\gtrsim 0.22$, D becomes smaller than D_{MF}.

The reason for these deviations is that the probability p_1^{eff} to find the walker at a 1 site is different from p_1. Consider a linear chain of finite but large length N with periodic boundary conditions. For $t\to\infty$ the probability $P_i(t)$ will then approach an equilibrium distribution P_i^{eq} which, according to Eq. (1), is determined by

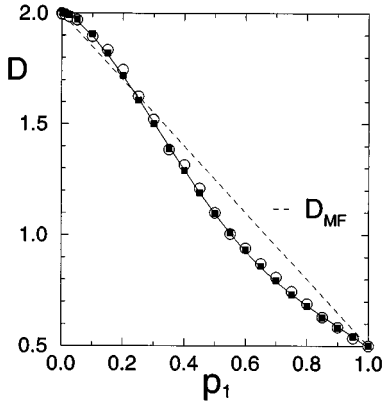

FIG. 2. Diffusion coefficient D determined from the mean-square displacement by Monte Carlo simulations (open circles) and calculated from p_1^{eff} [Eq. (4)] by applying the exact enumeration technique (solid squares). The dotted line indicates the mean-field result and the solid line is drawn as a guide for the eye.

$$P_i^{\mathrm{eq}}=\frac{1}{2}[(l_{i-2}-1)P_{i-2}^{\mathrm{eq}}+(2-l_{i-1})P_{i-1}^{\mathrm{eq}}+(2-l_{i+1})P_{i+1}^{\mathrm{eq}}+(l_{i+2}-1)P_{i+2}^{\mathrm{eq}}]. \tag{3}$$

By solving Eq. (3) one can calculate p_1^{eff} via $p_1^{\mathrm{eff}}=\langle\Sigma_{i=1}^{N}P_i^{\mathrm{eq}}(2-l_i)\rangle$, where $\langle\rangle$ denotes a disorder average over many chains with different configurations $\{l_i\}$. We determined P_i^{eq} (and thus p_1^{eff}) numerically by exact enumeration [8] of Eq. (1). By using p_1^{eff} in Eq. (2) instead of p_1, we get a diffusion coefficient that is shown by the solid squares in Fig. 2. Clearly, we find excellent agreement with the diffusion coefficient D, i.e.,

$$D=\frac{4-3p_1^{\mathrm{eff}}}{2}. \tag{4}$$

Hence we have to determine the effective distribution of jump lengths p_1^{eff} to obtain the correct diffusion coefficient. (We have found that the analogous result holds true for more general distributions of jump lengths p_l with finite second moment, i.e., $2D=\Sigma_l p_l^{\mathrm{eff}} l^2$.) As mentioned above, the linear relationship $\langle x^2(t)\rangle=2Dt$ is strictly valid only when the starting points of the walker are weighted with the correct equilibrium distribution. When the walker can start from each of the lattice sites with equal probability, the short-time behavior would be given by the mean-field result and only the long-time behavior would be determined by D.

In order to calculate p_1^{eff} analytically, we will use the following procedure. For a given configuration $\{l_i\}$, we subdivide the chain in clusters of 1 and 2 sites as in percolation theory [9]. A 1 cluster of size s consists of a row of s 1 sites with 2 sites at each end of the row. In an analogous manner a 2 cluster of size s is defined. The random walker consecutively enters 1 and 2 clusters of arbitrary size. One can calculate the average number $n_{j,s}(\alpha)$ ($j=1,2$, $\alpha=1,\ldots,s$) of sites visited, if a j cluster of size s is entered at site α. After averaging $n_{j,s}(\alpha)$ over all cluster sizes s and all possible initial positions α, one obtains the average number n_j of sites visited, when an *arbitrary* j cluster is entered. From this we obtain p_1^{eff} by

$$p_1^{\mathrm{eff}}=\frac{n_1}{n_1+n_2}. \tag{5}$$

In general, this procedure is difficult due to the existence of 1 clusters of size $s=1$, which we will call isolated 1 sites in the following. These isolated 1 sites can never be entered from a neighboring 2 cluster that consists of an odd number of sites. As a consequence, one has to merge certain 2 clusters into ''effective 2 clusters'' and the cluster statistics becomes complicated. However, one can apply a simpler approximate scheme, which we will present in Sec. IV. This scheme neglects the merging of 2 clusters and succeeds in reproducing the numerical p_1^{eff} values within an error of 2%, but it fails to predict the correct values for n_1 and n_2 separately. In Sec. V we then consider the general case taking into account the merging of 2-clusters by isolated 1-sites. This more elaborate treatment yields the correct dependence of n_1 and n_2 on p_1.

IV. APPROXIMATE TREATMENT

We need first to calculate the average number $n_{j,s}(\alpha)$ of sites visited, if a j cluster of size s is entered at site α. Let us start with the 1 clusters ($j=1$). The jumps of the particle inside a 1 cluster of size s can be considered as a random walk between absorbing points at site 0 and $s+1$. For this problem it is possible to calculate the conditional probability $\mathcal{P}_n(r|r_0;0,s+1)$ to find the walker at site r after n jumps, if it started at site r_0 ($r,r_0=1,\ldots,s$). By using the method of images we get [10]

$$\mathcal{P}_n(r|r_0;0,s+1)=\sum_{l=-\infty}^{\infty}[\boldsymbol{\mathcal{P}}_n(r+2l(s+1)|r_0) - \boldsymbol{\mathcal{P}}_n(-r-2l(s+1)|r_0)], \tag{6}$$

where

$$\boldsymbol{\mathcal{P}}_n(r|r_0)=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\cos^n(k)e^{ik(r-r_0)} \tag{7}$$

is the conditional probability when no absorbing points are present. With the help of

$$\sum_{l=-\infty}^{\infty}\exp[-2il(s+1)k]=\frac{\pi}{s+1}\sum_{l=-\infty}^{\infty}\delta\left(k-\frac{\pi l}{s+1}\right), \tag{8}$$

we obtain after simple transformations

$$\mathcal{P}_n(r|r_0;0,s+1) = \frac{2}{s+1}\sum_{l=1}^{s}\cos^n\left(\frac{\pi}{s+1}l\right)\sin\left(\frac{\pi r_0}{s+1}l\right)\sin\left(\frac{\pi r}{s+1}l\right). \tag{9}$$

The probability $\psi_s(n|r_0)$ for the walker to make exactly n jumps inside a cluster of size s, when it started at site r_0, is given by

$$\psi_s(n|r_0)=\frac{1}{2}[\mathcal{P}_n(1|r_0;0,s+1)+\mathcal{P}_n(s|r_0;0,s+1)]. \tag{10}$$

Hence we finally obtain, for $n_s(\alpha)\equiv n_{1,s}(\alpha)$,

$$n_s(\alpha)=\sum_{n=0}^{\infty}(1+n)\psi_s(n|\alpha)=(s+1)\alpha-\alpha^2,\quad 1\leqslant\alpha\leqslant s. \tag{11}$$

(Note that we averaged $1+n$ and not n since the initial site has to be counted too.)

The average number $n_{2,s}(\alpha)$ of sites visited, if a 2 cluster of size s is entered at site α can be readily calculated from $n_s(\alpha)$ since the random walk inside a 2 cluster corresponds to a random walk inside a 1 cluster of half size (by taking out all 2 sites that cannot be visited). Depending on the size s of the 2 cluster and the initial site α, we obtain

$$n_{2,s}(\alpha)=\begin{cases} n_{(s+1)/2}\left(\dfrac{\alpha+1}{2}\right) & \text{for } s \text{ odd}, \quad \alpha \text{ odd} \\ n_{(s-1)/2}\left(\dfrac{\alpha}{2}\right) & \text{for } s \text{ odd}, \quad \alpha \text{ even} \\ n_{s/2}\left(\dfrac{\alpha+1}{2}\right) & \text{for } s \text{ even}, \quad \alpha \text{ odd} \\ n_{s/2}\left(\dfrac{\alpha}{2}\right) & \text{for } s \text{ even}, \quad \alpha \text{ even}. \end{cases} \tag{12}$$

In order to calculate n_1 and n_2 one now has to average $n_{1,s}(\alpha)$ and $n_{2,s}(\alpha)$ over all cluster sizes s and all possible initial conditions α. Following the cluster analysis in [9], the mean total number $m_{j,s}$ of j clusters with s sites on a long chain of length N is $m_{j,s}=Np_j^s(1-p_j)^2$. The probability $w_{j,s}$ that a j cluster consists of s sites is then

$$w_{j,s}=\frac{m_{j,s}}{\sum_{s=1}^{\infty}m_{j,s}}=(1-p_j)p_j^{s-1}. \tag{13}$$

A 2 cluster can only be entered at a boundary site, so we can set $\alpha=1$ [due to symmetry the left and right boundary site must not be distinguished, $n_{2,s}(1)=n_{2,s}(s)$]. Hence we obtain

$$n_2=\sum_{s=1}^{\infty}w_{2,s}n_{2,s}(1)=\sum_{s=1}^{\infty}(w_{2,2s-1}+w_{2,2s})n_s(1) = 1+\frac{p_2^2}{p_1(1+p_2)}. \tag{14}$$

A 1 cluster can be entered (a) at a boundary site ($\alpha=1$) or (b) at the next neighbor of a boundary site ($\alpha=2$). Case (a) can occur only if the walker before has entered a 2 cluster with an even number of sites, which we call an ''even 2 cluster'' (in the same way an ''odd 2 cluster'' is defined). The walker has to enter the even 2 cluster from one side (say the left boundary site) and to escape it from the other side (the right boundary site). Case (b) takes place when the walker before has entered an arbitrary (even or odd) 2 cluster from the left boundary site and escaped it from the same left boundary site or when the walker before has entered an odd 2 cluster from the left boundary site and escaped it from the opposite right boundary site. We denote the probability for event (a) by π_a and that for event (b) by π_b. Given π_a and π_b one may calculate n_1,

$$n_1=\pi_a\sum_{s=1}^{\infty}w_{1,s}n_s(1)+\pi_b\sum_{s=2}^{\infty}w_{1,s}n_s(2) = \pi_a p_2+\sum_{s=2}^{\infty}w_{1,s}[\pi_a n_s(1)+\pi_b n_s(2)]. \tag{15}$$

Note that the second series starts from $s=2$ and not from $s=1$ since an isolated 1 site can never be entered in case (b). However, this method to treat the effect of isolated 1 sites is only approximate. When an isolated 1 site is not entered, it is actually overjumped and the 2 clusters on its left and right

side have to be regarded as one large 2 cluster. This merging of 2 clusters due to the presence of isolated 1 sites has been fully neglected.

Our approximate solution is complete if we know π_a and π_b. Similarly as for $n_{2,s}(\alpha)$ above, we first calculate the relevant escape probabilities (''left'' or ''right'') for the 1 clusters and then determine the analogous probabilities for the 2 clusters by treating the random walk inside a 2 cluster as being inside a 1 cluster of half size. The probability $L_s(\alpha)$ that the random walker leaves a 1 cluster of size s via the left boundary site, if it was initially at site α, is

$$L_s(\alpha)=\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{P}_n(1|\alpha;0,s+1)=1-\frac{\alpha}{s+1} \tag{16}$$

and the corresponding probability $R_s(\alpha)$ that it leaves a 1 cluster of size s via the right boundary site is

$$R_s(\alpha)=\frac{1}{2}\sum_{n=0}^{\infty}\mathcal{P}_n(s|\alpha;0,s+1)=1-L_s(\alpha)=\frac{\alpha}{s+1}. \tag{17}$$

The probability π_a is determined by averaging $R_s(1)$ over all *even* 2 clusters,

$$\pi_a=\sum_{s=1}^{\infty}w_{2,2s}R_s(1)=p_1\sum_{s=1}^{\infty}\frac{p_2^{2s-1}}{s+1}=\frac{p_1}{p_2^3}\ln\left(\frac{e^{-p_2^2}}{1-p_2^2}\right). \tag{18}$$

In the same way we obtain

$$\pi_b=\sum_{s=1}^{\infty}(w_{2,2s-1}+w_{2,2s})L_s(1)+\sum_{s=1}^{\infty}w_{2,2s-1}R_s(1)$$
$$=1-\pi_a. \tag{19}$$

With π_a given by Eq. (18), it is easy to calculate n_1 from Eq. (15),

$$n_1=\left(1-\frac{p_1}{p_2}\right)\pi_a+2\frac{p_1}{p_2}, \tag{20}$$

and finally $p_1^{\rm eff}$ according to Eq. (5),

$$p_1^{\rm eff}=\frac{2p_1+\frac{p_1(p_2-p_1)}{p_2^3}\ln\left(\frac{e^{-p_2^2}}{1-p_2^2}\right)}{1+p_1+\frac{p_2^3}{p_1(1+p_2)}+\frac{p_1(p_2-p_1)}{p_2^3}\ln\left(\frac{e^{-p_2^2}}{1-p_2^2}\right)}. \tag{21}$$

Figure 3(a) shows $p_1^{\rm eff}$ as a function of p_1 (solid line) in comparison with the values obtained from the exact enumeration technique (solid squares). As can be seen from the figure, the agreement between the approximate theory and the ''exact'' numerics is very good; the data do not deviate by more than 2%. For $p_1\to 0$, Eq. (21) predicts a nonanalytic dependence of $p_1^{\rm eff}$ on p_1, $p_1^{\rm eff}\sim -2p_1^2\ln p_1$, and indeed the data are in nice agreement with this prediction [see the inset of Fig. 3(a)]. However, as shown in Fig. 3(b), the separate dependence of n_1 and n_2 on p_1 is not correctly described

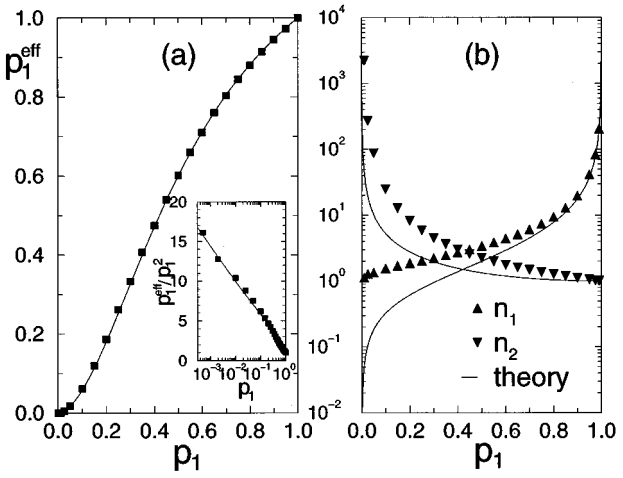

FIG. 3. (a) Probability $p_1^{\rm eff}$ obtained from the approximate theoretical treatment [Eq. (21)] as a function of p_1 (solid line) in comparison with the results from the exact enumeration of Eq. (1) (solid squares). The inset shows the behavior of $p_1^{\rm eff}/p_1^2$ as a function of p_1 in a semilogarithmic representation. (b) Mean number n_1 and n_2 of sites visited, when the walker enters an arbitrary 1 and 2 cluster. The symbols mark the results from Monte Carlo simulations and the solid lines the results from the approximate theory [Eqs. (15) and (14)].

(the data points in the figure were obtained by Monte Carlo simulations). Except for p_1 close to 1, the approximate theory yields much smaller values for both n_1 and n_2 than the Monte Carlo simulations. This failure is due to the incorrect treatment of the overjumping and associated merging effect: Under the condition that an isolated 1 site is entered, the number of visited sites is always one but not π_a, as implicitly assumed in Eq. (15). Hence n_1 should approach one for $p_1\to 0$ and not zero as predicted by Eq. (15). On the other hand, the 2 clusters are effectively larger and accordingly also n_2 is underestimated by the approximate theory. Since both effects are mutually connected, the errors in n_1 and n_2 approximately cancel each other in the ratio n_1/n_2, which determines the probability $p_1^{\rm eff}$.

V. GENERAL TREATMENT

We will now take into account the merging of ''geometric'' 2 clusters into ''effective'' 2 clusters caused by isolated 1 sites that cannot be visited. The definition of these effective 2 clusters depends on the way they are entered. More precisely, one has to distinguish between two situations: (i) The walker enters an effective 2 cluster from a 1 cluster with $s\geqslant 2$ and (ii) the walker enters an effective 2 cluster from an isolated 1 site.

Let us start with case (i). Without loss of generality, we can assume that the walker enters the effective 2 cluster from the left. Then we can describe the general situation as in Fig. 4(a): The encircled plus denotes the 1 cluster, from which the walker enters the effective 2 cluster. The encircled dot denotes isolated 1 sites, which can be overjumped. They have odd 2-clusters as neighbors consisting of $2\gamma_i-1$ sites ($i=0,1,\ldots,k$, and $\gamma_i=1,2,\ldots$). The encircled cross marks the right end of the effective 2 cluster. This can be either an

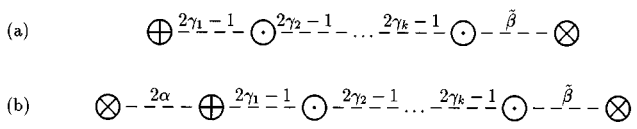

FIG. 4. General configurations (a) if an effective 2 cluster is entered from a 1 cluster of size larger than one and (b) if an effective 2 cluster is entered from an isolated 1 site. The $\odot$ symbols mark isolated 1 sites, which can be overjumped, the $\oplus$ symbol denotes the 1 cluster, from which the effective 2 cluster is entered [1 cluster with $s \geq 2$ in (a) and isolated 1 site in (b)]; and the $\otimes$ symbols mark the ends of the effective 2 cluster. The dashed lines represent the geometric 2 clusters between the 1 clusters and the numbers $(2\alpha, 2\gamma_1 - 1, \ldots)$ refer to their sizes.

arbitrary 1 cluster if the size $\tilde{\beta}$ of the neighboring geometric 2 cluster is even or it must be a 1 cluster of size larger than one if $\tilde{\beta}$ is odd. The number of *distinct* sites s that can be visited inside the effective 2 cluster is $s = \beta + \Sigma_{i=1}^{k} \gamma_i$, where $2\beta - 1 = \tilde{\beta}$ if $\tilde{\beta}$ is odd and $2\beta = \tilde{\beta}$ if $\tilde{\beta}$ is even ($\beta = [(1+\tilde{\beta})/2]$, where $[x]$ denotes the integer part of x). It is convenient to define this number s of distinct visitable sites as the size of the effective 2 cluster. In this way we avoid later having to take out formally all nonvisitable sites, as it was done in Sec. IV in connection with the geometric 2 clusters.

According to the general situation shown in Fig. 4(a), we can calculate the probability $w_{2,s}^{(1)}$ that if an effective 2 cluster is visited, it has size s and is entered from a 1 cluster of size larger than one:

$$\begin{aligned} Z w_{2,s}^{(1)} &= p_1^2 (p_2^{2s-1} p_1^2 + p_2^{2s} p_1) \\ &\quad + \sum_{\gamma_1 + \beta = s} p_1^2 p_2^{2\gamma_1 - 1} p_1 (p_2^{2\beta-1} p_1^2 + p_2^{2\beta} p_1) \\ &\quad + \sum_{\gamma_1 + \gamma_2 + \beta = s} p_1^2 p_2^{2\gamma_1 - 1} p_1 p_2^{2\gamma_2 - 1} p_1 (p_2^{2\beta-1} p_1^2 \\ &\quad + p_2^{2\beta} p_1) + \cdots \\ &= p_1^3 p_2^{2s-1} \left[1 + \left(\frac{p_1}{p_2}\right) A(s,2) + \left(\frac{p_1}{p_2}\right)^2 A(s,3) + \cdots \right. \\ &\quad \left. + \left(\frac{p_1}{p_2}\right)^{s-1} A(s,s) \right] \\ &= p_1^3 p_2^{2s-1} \sum_{k=0}^{s-1} A(s,k+1) \left(\frac{p_1}{p_2}\right)^k . \end{aligned} \tag{22}$$

Here Z is a normalization factor that will be determined below. The two terms in the factor $(p_2^{2\beta-1} p_1^2 + p_2^{2\beta} p_1)$ in the second line of Eq. (22) correspond to the two possibilities that the rightmost geometric 2 cluster consists of an odd ($\tilde{\beta} = 2\beta - 1$) or even number ($\tilde{\beta} = 2\beta$) of sites. The combinatorial factor $A(s,k)$ equals the number of ways to add up k integer numbers larger than 0 to a sum s, when taking into account their ordering. In other words, $A(s,k)$ is the number of ways to distribute s identical (i.e., nondistinguishable) particles among k distinguishable states given the constraint that each state is at least occupied by one particle. As shown in Appendix, $A(s,k)$ can be determined in the spirit of this slightly modified Bose statistics with the result

$$A(s,k) = \binom{s-1}{k-1} . \tag{23}$$

Accordingly, the final sum in Eq. (22) is a binomial series and can be calculated explicitly

$$w_{2,s}^{(1)} = \frac{p_1^3 p_2^s}{Z} . \tag{24}$$

Next we consider case (ii), i.e., the walker enters an effective 2 cluster from an isolated 1 site. An isolated 1 site can be visited only if one of its neighboring geometric 2 clusters consists of an even number of sites. Without loss of generality, we assume that the left neighbor is an even geometric 2 cluster. The general situation can then be described as in Fig. 4(b): Again, the encircled cross marks the ends of the effective 2 cluster, the encircled dot the isolated 1 sites, which can be overjumped, and the encircled plus denotes the now isolated 1 site, from which the effective 2 cluster is entered. The size s of the effective 2 cluster, i.e., the number of distinct visitable sites inside it, is $s = \alpha + \beta + \Sigma_{i=1}^{k} \gamma_i$, where $\beta = [(1+\tilde{\beta})/2]$.

We now can calculate the probability $w_{2,s}^{(2)}(\alpha)$ that if an effective 2 cluster is visited, it has size s and is entered form an isolated 1 site, which has an even geometric 2 cluster consisting of 2α sites as neighbor:

$$\begin{aligned} Z w_{2,s}^{(2)}(\alpha) &= \sum_{\beta + \gamma_1 + \cdots + \gamma_k = s - \alpha} v_\alpha p_1 p_2^{2\alpha} p_1 p_2^{2\gamma_1 - 1} \cdots \\ &\quad \times p_2^{2\gamma_k - 1} p_1 (p_2^{2\beta-1} p_1^2 + p_2^{2\beta} p_1) \\ &= v_\alpha p_1^3 p_2^{2\alpha - 1} \left[p_2^{2(s-\alpha)} + \sum_{\beta + \gamma_1 = s - \alpha} p_2^{2\beta} \right. \\ &\quad \left. \times p_2^{2\gamma_1 - 1} p_1 + \cdots \right] \\ &= v_\alpha p_1^3 p_2^{2s-1} \sum_{k=0}^{s-1-\alpha} \binom{s-1-\alpha}{k} \left(\frac{p_1}{p_2}\right)^k \\ &= v_\alpha p_1^3 p_2^{s+\alpha} . \end{aligned} \tag{25}$$

The factor v_α takes into account the relative weight according to which the effective 2 cluster is entered from the isolated 1 site. It is possible to determine these factors v_α in a self-consistent way, but then the calculations become very lengthy and complicated. It is advantageous to make an approximation and to assume that for the effective 2 cluster to be entered from the isolated 1 site, the walker must before have entered the neighboring geometric 2 cluster with 2α sites from the ''left'' end and must then have escaped this cluster via the ''right'' end. This implies $v_\alpha \simeq R_\alpha(1) = 1/(\alpha+1)$ and, accordingly,

$$w_{2,s}^{(2)}(\alpha) = \frac{p_1^3 p_2^{s+\alpha}}{Z(\alpha+1)} . \tag{26}$$

The normalization factor Z is given by

$$Z=\sum_{s=1}^{\infty} p_1^3 p_2^s+\sum_{s=2}^{\infty}\sum_{\alpha=1}^{s-1}\frac{p_1^3 p_2^{s+\alpha}}{\alpha+1}$$

$$=p_1^2 p_2+p_1^3\sum_{s=2}^{\infty} p_2^s\sum_{\alpha=1}^{s-1}\frac{p_2^\alpha}{\alpha+1}. \tag{27}$$

The double sum over s and α can be calculated after interchanging the summation order. The resulting expression is

$$Z=-\frac{p_1^2}{p_2}\ln(1-p_2^2). \tag{28}$$

Similar double sums are encountered in various formulas below and they are calculated in an analogous way.

Knowing the cluster distribution of the effective 2 clusters, we can determine the average number n_2 of sites visited if an arbitrary 2 cluster is entered:

$$n_2=\sum_{s=1}^{\infty} w_{2,s}^{(1)} n_s(1)+\sum_{s=2}^{\infty}\sum_{\alpha=1}^{s-1} w_{2,s}^{(2)}(\alpha)\tilde{n}_s(\alpha)$$

$$=-\frac{1+p_1}{2p_1}-\frac{p_2^2}{2p_1\ln(1-p_2^2)}\left[3+p_1+\frac{p_2^2(4-p_1^2)}{p_1(1+p_2)^2}\right]. \tag{29}$$

Here we have defined

$$\tilde{n}_s(\alpha)=\frac{1}{2}[n_s(\alpha)+n_s(\alpha+1)], \tag{30}$$

where the two terms refer to the two possibilities that the walker enters an effective 2 cluster from an isolated 1 site by a jump to the left or by a jump to the right. We will use the tilde in a similar manner below.

Next, we calculate the new cluster distribution for the 1 clusters. If an arbitrary 1 cluster is visited, the probability $w_{1,1}^{(0)}$ that it has size $s=1$ (isolated 1 site) is

$$w_{1,1}^{(0)}=\sum_{s=1}^{\infty} w_{2,s}^{(1)} R_s(1)p_2^2$$

$$+\sum_{s=2}^{\infty}\sum_{\alpha=1}^{s-1} w_{2,s}^{(2)}(\alpha)[\tilde{L}_s(\alpha)p_2+\tilde{R}_s(\alpha)p_2^2]$$

$$=1+\frac{p_2^2}{\ln(1-p_2^2)}-\frac{p_1^2}{4\ln(1-p_2^2)}$$

$$\times[(p_2+\ln p_1)^2-g_2(p_2^2)+p_2^2], \tag{31}$$

where $\tilde{L}_s(\alpha)=[L_s(\alpha)+L_s(\alpha+1)]/2$, $\tilde{R}_s(\alpha)=[R_s(\alpha)+R_s(\alpha+1)]/2=1-\tilde{L}_s(\alpha)$, and $g_2(x)=\Sigma_{k=1}^{\infty}x^k/k^2$. The three different terms in the first line of Eq. (31) refer to the following cases. (i) The walker comes from an effective 2 cluster that itself was entered before from a 1 cluster of size larger than one (hence the factor $w_{2,s}^{(1)}$). For the walker to arrive at an isolated 1 site, it must have entered and escaped this effective 2 cluster at opposite boundary sites [hence the factor $R_s(1)$]. In particular, the walker had to leave the effective 2 cluster via an even geometric boundary cluster ($\tilde{\beta}$ even in the configuration shown above) and really to arrive at an isolated 1 site [hence the factor $p_2^{2\beta}p_1p_2/(p_2^{2\beta}p_1+p_2^{2\beta-1}p_1^2)=p_2^2$]. (ii) The walker comes from an effective 2 cluster that itself was entered before from an isolated 1 site whose neighboring even geometric 2 cluster had 2α sites [factor $w_{2,s}^{(2)}(\alpha)$]. It escaped the effective 2 cluster via this even geometric 2 cluster [factor $\tilde{L}_s(\alpha)$] and it had to arrive at an isolated 1 site (factor p_2). (iii) As in (ii), the walker comes from an effective 2 cluster that itself was entered from an isolated 1 site [factor $w_{2,s}^{(2)}(\alpha)$]. But it escaped this effective 2 cluster on the other side [factor $\tilde{R}_s(\alpha)$] and via an even geometric 2 cluster in order to arrive at an isolated 1 site [same factor p_2^2 as in case (i)].

The probability $w_{1,s}^{(0)}$ that a 1 cluster of size larger than one consists of exactly s sites is

$$w_{1,s}^{(0)}=p_1^{s-2}p_2. \tag{32}$$

Again we have to distinguish between case (a), where a 1 cluster is entered at a boundary site, and case (b), where it is entered at the next neighbor of a boundary site. Event (a) can occur in all three cases, which contributed to $w_{1,1}^{(0)}$ above, with the difference that now the three terms have to be multiplied by p_1 instead of p_2 in order to count events only where the walker arrives at a 1 cluster of size larger than one. Hence we obtain for the probability π_a

$$\pi_a=\frac{p_1}{p_2}w_{1,1}^{(0)}=p_1p_2\sum_{s=1}^{\infty} w_{2,s}^{(1)}R_s(1)+p_1\sum_{s=2}^{\infty}\sum_{\alpha=1}^{s-1} w_{2,s}^{(2)}(\alpha)$$

$$\times[\tilde{L}_s(\alpha)+p_2\tilde{R}_s(\alpha)]. \tag{33}$$

The probability π_b can be calculated analogously to π_a (or $w_{1,1}^{(0)}$):

$$\pi_b=\sum_{s=1}^{\infty} w_{2,s}^{(1)}[L_s(1)+p_1R_s(1)]$$

$$+p_1\sum_{s=2}^{\infty}\sum_{\alpha=1}^{s-1} w_{2,s}^{(2)}(\alpha)\tilde{R}_s(\alpha). \tag{34}$$

One can easily verify that the cluster distribution for the 1 clusters is correctly normalized,

$$w_{1,1}^{(0)}+(\pi_a+\pi_b)\sum_{s=2}^{\infty} w_{1,s}^{(0)}=1. \tag{35}$$

Finally, having determined the new cluster distribution for the 1 clusters and the entrance probabilities π_a and π_b, we can calculate the mean number n_1 of sites visited when the walker enters an arbitrary 1 cluster:

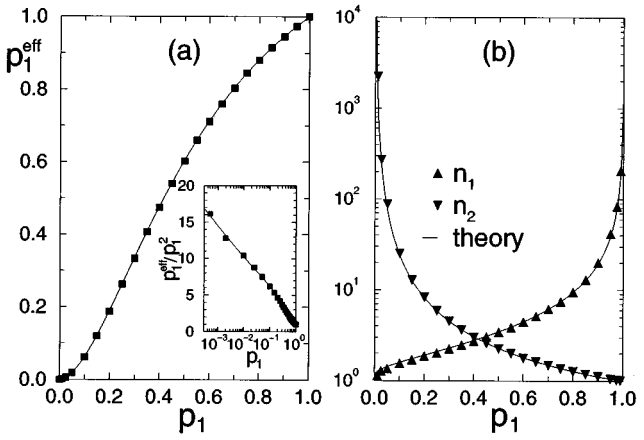

FIG. 5. (a) Probability p_1^{eff} obtained from the general treatment as a function of p_1 (solid line) in comparison with the results from the exact enumeration of Eq. (1). The inset shows the dependence of p_1^{eff}/p_1^2 on p_1 in a semilogarithmic representation. (b) Mean number n_1 and n_2 of sites visited, when the walker enters an arbitrary 1 and 2 cluster. The symbols mark the results from Monte Carlo simulations and the solid lines the results from the general theory [Eqs. (36) and (29)].

$$n_1 = w_{1,1}^{(0)} + \sum_{s=2}^{\infty} w_{1,s}^{(0)} [\pi_a n_s(1) + \pi_b n_s(2)]$$

$$= \frac{p_2 - p_1}{p_2^2} + \frac{1}{\ln(1-p_2^2)} \times \left(\frac{p_1^2}{4p_2^2} [(p_2 + \ln p_1)^2 - g_2(p_2^2) + p_2^2] - 1 \right). \quad (36)$$

Figure 5 shows (a) $p_1^{\text{eff}} = n_1/(n_1+n_2)$ and (b) n_1 and n_2 as functions of p_1 according to the theory (solid line) in comparison with the same numerical results shown in Fig. 3. The agreement between theory and numerics is now excellent, also for n_1 and n_2. Remarkably, in the limit $p_1 \to 0$, the general theory predicts the same nonanalytical dependence of p_1^{eff} on p_1, $p_1^{\text{eff}} \sim -2p_1^2 \ln p_1$ [see the inset of Fig. 5(b)]. In the opposite limit $p_1 \to 1$ ($p_2 \to 0$), we obtain $n_1 \sim 2/p_2 - 1/2$ and $n_2 \sim 1 + p_2$, while in the approximate treatment $n_1 \sim 2/p_2 - 5/2$ and $n_2 \sim 1 + p_2^2$. Hence, even when the number of isolated 1 sites becomes small, the consequences of the overjumping and merging effect appear already in the first corrections to the leading terms.

VI. CONCLUSION

In this work we have studied the consequences of a quenched jump length distribution on the diffusion properties of a single random walker. We have focused on the simplest case, i.e., the random walk on a linear chain with a bimodal distribution of jump lengths 1 and 2. Two competing effects lead to pronounced deviations from the annealed case (mean-field result): (i) the longer residence time of the walker in 1 clusters than in 2 clusters (of the same size) and (ii) the overjumping of isolated 1 sites. The overjumping effect dominates the behavior for small concentrations p_1 of 1 sites and leads to an enhancement of the diffusion coefficient relative to the mean-field result. For larger values of p_1 ($p_1 \gtrsim 0.22$), the overjumping effect is less important and effect (i) causes the walker to diffuse more slowly than in the annealed case.

It is clear that the effects (i) and (ii) are essential also when considering more general distributions p_l with $l = 1,2,3, \ldots$. Although the analytical calculations in Secs. IV and V then become more difficult to perform, one can expect that the diffusion coefficient in the quenched case is usually smaller than in the annealed case, except for those distributions p_l that give the short jump lengths l a comparatively small weight. The same effects will be also present for random walks in higher dimensions d. However, since the surface-to-volume ratio of l clusters with a given size increases with d, the random walker can more easily ''escape'' the fluctuations. We thus expect the differences between the quenched and annealed case to become less pronounced for higher d.

An interesting situation arises when the distribution p_l has no finite second moment as for a Lévy distribution, $p_l \sim l^{-1-f}$ with $0 < f < 2$. Preliminary results [11] show that the effective distribution p_l^{eff} decays in a different way than p_l, $p_l^{\text{eff}} \sim l^{-1-g}$, with an exponent $g \geq f$. As a consequence, the ''superdiffusive'' behavior is slowed down in comparison to the annealed case. This might have important consequences for the description of various phenomena by Lévy walks [12].

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APPENDIX: PROOF THAT $A(s,k) = \binom{s-1}{k-1}$

As stated in Sec. V above, $A(s,k)$ equals the number of ways to distribute s identical particles among k distinguishable states given the constraint that each state is at least occupied by one particle. For $k=1$ there is only one way: All particles have to be put into the single state. Hence $A(s,1) = 1$, which agrees with $\binom{s-1}{0} = 1$.

We now show by complete induction after k (for arbitrary $s \geq k$) that Eq. (23) is valid. Let us therefore assume that $A(s,k) = \binom{s-1}{k-1}$ for $s \geq k+1$. If we now add one state, we can put either $1,2, \ldots, s-k-1$ or $s-k$ particles into this additional state. For the remaining k states there are then $s-1, s-2, \ldots, k+1$ or k particles left. Hence we obtain

$$A(s,k+1) = A(s-1,k) + A(s-2,k) + \cdots + A(k,k) = \sum_{j=0}^{s-k-1} A(j+k,k) = \sum_{j=0}^{s-k-1} \binom{j+k-1}{k-1} = \binom{s-k+k-1}{k-1+1} = \binom{s-1}{k}, \quad (A1)$$

which completes the proof.

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